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Complex time path for tunnelling at intermediate energy

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Abstract. We investigate tunnelling through a potential barrier in quantum mechanics extending Feynman's path integral to complex time paths. This problem has gained new interest in the context of baryon number violating processes in the standard electroweak model, where the transitions between topological sectors must be described at high energies and temperatures. In this paper we reduce the number of degrees of freedom to one in order to test our method. It is found that a path in the complex time plane consisting of purely real or imaginary steps minimizes the real and imaginary parts of the action simultaneously. Jumps from real to imaginary time increments are situated at the turning points of the classical problem. Possible extensions to field theory are discussed.

1. Introduction

Tunnelling processes belong to the oldest miracles in quantum physics. Recently tunnelling has gained new interest which stems from the calculation of high energetic scattering processes in the framework of the standard electroweak model [1, 2]. It has soon become clear that the naive estimates suffer from an inability to reconcile different viewpoints about the mechanism of transition between two distinct topological sectors, at energies comparable to the so-called sphaleron energy. This concerns the simultaneous understanding of the process from the Euclidean (instanton) and the Minkowskian (sphaleron) perspectives, respectively. It is obvious that the zero-energy O(4)-symmetric instanton solution is not a suitable background for high-energy scattering involving baryon number violation. Various improvements have been suggested but none of them has proved to solve the problem satisfactorily. One interesting proposal has been given by Khlebnikov *et al* [3]. They discuss the concept of using (parts of) periodic instanton solutions in instanton-induced processes. Their conclusion was pessimistic with respect to the expected cross section of baryon-violating processes. Nevertheless we think that their idea of a path-integral description combining Minkowski and Euclidean points of view is worth discussing in more detail.

In this paper we illustrate a numerical procedure by applying it to a quantum mechanical toy model for barrier penetration, whose exact solution is known and can be confronted with the result of our complex time-path method. There is one principal difference between the toy and the real problem: in the case of one degree of freedom there are exactly calculable turning points (of the classical trajectory) while there are none in field theory. This circumstance is extremely helpful in the toy case, but could cause certain difficulties when one wants to apply our method to field-theoretic models. We will discuss this aspect at the end of our paper.

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2. The complex time-path method from Feynman's path integral

The investigation of tunnelling processes in the framework of path integrals and the concept of complex time paths was discussed in the seventies, e.g. by McLaughlin [4] and Miller [5], and later by Carlitz and Nicole [6]. In the present paper it is the principle of extremal (real and imaginary) parts of the reduced action, postulated in the context of the path integral, that we would like to put into the first position. This principle could serve both as a motivation and as a constructive tool for practising complex time-path methods in general.

We consider a simple model of a particle with energy E in the field of a potential V, described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}m\dot{q}^2 - V(q)\,. \tag{1}$$

In quantum mechanics the transition amplitude from a state $q_{\rm in}, t_{\rm in}$ to a state $q_{\rm fi}, t_{\rm fi}$ is given by the expression

$$W(q_{\rm fi}, t_{\rm fi}; q_{\rm in}, t_{\rm in}) = \langle q_{\rm fi}, t_{\rm fi} \mid q_{\rm in}, t_{\rm in}\rangle = \int_{q_{\rm in}t_{\rm in}}^{q_{\rm fi}t_{\rm fi}} \mathrm{d}q(\tau)\, \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_{\rm in}}^{t_{\rm fi}} \mathcal{L}\,\mathrm{d}\tau\right)\,. \tag{2}$$

Expanding around the classical path $q_{\rm cl}(t)$,

$$q(t) = q_{\rm cl}(t) + y(t)$$

one gets the following perturbative representation for W:

$$W(q_{\rm fi}, t_{\rm fi}; q_{\rm in}, t_{\rm in}) = \exp\left(\frac{\mathrm{i}}{\hbar}S_{\rm cl}(q_{\rm fi}, t_{\rm fi}; q_{\rm in}, t_{\rm in})\right)\, \tilde{W}(0, t_{\rm fi}; 0, t_{\rm in}) \tag{3}$$

where up to the second order in $y(t)$ (i.e. to one-loop accuracy) $\tilde{W}$ is approximated by

$$\tilde{W}(0, t_{\rm fi}; 0, t_{\rm in}) \approx \int_{0t_{\rm in}}^{0t_{\rm fi}} \mathrm{d}y(\tau) \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_{\rm in}}^{t_{\rm fi}} \frac{1}{2}\frac{\delta^2 S}{\delta q^2}\bigg|_{q=q_{\rm cl}} y(\tau)^2\,\mathrm{d}\tau\right)\,. \tag{4}$$

In leading order only the first exponential $\exp[(\mathrm{i}/\hbar)S_{\rm cl}]$ is taken into account. For a particle with fixed energy E the Fourier transform of W is the appropriate quantity to consider (we set $t_{\rm in} = 0$ and $t_{\rm fi} = T$ without loss of generality):

$$G(q_{\rm fi}, q_{\rm in}; E) = \int_{\rm in}^{\infty} \exp\left(\frac{\mathrm{i}\,E\,T}{\hbar}\right) W(q_{\rm fi}, T; q_{\rm in}, 0)\,\mathrm{d}T\,. \tag{5}$$

Inserting the WKB approximation for W in the Fourier transform and defining the so-called reduced action,

$$\Gamma = E\,T + S_{\rm cl} \tag{6}$$

the following expression is obtained:

$$G_{\rm WKB}(q_{\rm fi}, q_{\rm in}; E) = \mathrm{i}\left(\frac{\mathrm{i}}{2\pi\hbar}\right)^{-\frac{1}{2}} \sum_{\text{class paths}} \sqrt{|\tilde{D}|}\,\exp\left(\frac{-\mathrm{i}n\pi}{2} + \frac{\mathrm{i}}{\hbar}\tilde{\Gamma}\right) \tag{7}$$

where $\tilde{\Gamma}$ has an E-dependence which results from the stationary phase condition

$$\frac{\partial}{\partial T}\Gamma = 0 \Rightarrow \tilde{T} \Rightarrow \tilde{\Gamma} = E\tilde{T} + S_{\mathrm{cl}}(q_{\mathrm{fi}}, q_{\mathrm{in}}; \tilde{T}) .$$

The integer n is the number of turning points along the classical path, and $\tilde{D}$ is defined for a one-dimensional problem

$$\tilde{D} = \frac{\partial^2 S_{\mathrm{cl}}/\partial q_{\mathrm{fi}}\partial q_{\mathrm{in}}}{\partial^2 S_{\mathrm{cl}}/\partial T^2} = \det \begin{pmatrix} \partial^2\Gamma/\partial q_{\mathrm{in}}\partial q_{\mathrm{fi}} & \partial^2\Gamma/\partial q_{\mathrm{in}}\partial E \\ \partial^2\Gamma/\partial q_{\mathrm{fi}}\partial E & \partial^2\Gamma/\partial E^2 \end{pmatrix} . \tag{8}$$

Equation (7) describes classically allowed trajectories. Things become more complicated when the energy of a particle is less than the height of a barrier placed on its classical path. Then a classical solution does not exist everywhere and (7) cannot be applied in a straightforward manner. However, analytic continuation of the time variable into the complex plane can save the situation [7, 8]. McLaughlin [4] has arrived at the following generalization:

$$G(q_{\mathrm{fi}}, q_{\mathrm{in}}; E + \mathrm{i}\epsilon) = \mathrm{e}^{-3\pi\mathrm{i}/4}\left[\frac{m}{2}\left(\frac{\partial^2 S_{\mathrm{cl}}}{\partial q_{\mathrm{fi}}\partial q_{\mathrm{in}}}\right)\left(\frac{\partial^2 S_{\mathrm{cl}}}{\partial \tau_0^2}\right)^{-1}\right]^{\frac{1}{2}} \times \exp\left(\frac{\mathrm{i}}{\hbar}[(E + \mathrm{i}\epsilon)\tau_0 + S_{\mathrm{cl}})]\right) \tag{9}$$

where τ_0 is the so called *critical point* given by the relation

$$\tau_0 = \sqrt{\frac{m}{2}} \int_{q_{\mathrm{in}}}^{q_{\mathrm{fi}}} \mathrm{d}z\,[\xi - V(z)]^{-\frac{1}{2}} \qquad \xi = E + \mathrm{i}\epsilon \qquad \epsilon > 0 . \tag{10}$$

Miller [5] started from a classical trajectory in a given potential, e.g.

$$x(t) = -\sinh^{-1}[(\alpha/E - 1)^{-\frac{1}{2}}\cosh(vt)] \tag{11}$$

for a particle with energy E (and velocity v in the asymptotical past) under reflection from the Eckart potential wall ((32), to be discussed below). He identified a set of branching points in the complex time plane at

$$t_n = \pm t_\star \pm \frac{\mathrm{i}(n + 0.5)\pi}{v} \qquad t_\star = \frac{1}{v}\tanh^{-1}[(E/\alpha)^{-\frac{1}{2}}]$$

which were then used to classify possible trajectories approaching $x \to \pm\infty$ at infinite time, after some amount of imaginary time 'spent' under the barrier. Miller [5] was interested mainly in the leading WKB approximation of the amplitudes, to be summed over topologically equivalent trajectories.

In this paper we follow another strategy to obtain the solution: for a given energy E we look for the trajectory minimizing Γ, aiming to obtain simultaneously a series of real coordinates q_i and a series of complex times t_i. Also, we will extend a well known method to calculate the determinant to this unconventional space and time trajectory. In figure 1 the situation for a single barrier penetration is shown. Regions I and III contain the classically allowed paths whereas region II is forbidden classically. A particle coming from region I to end up in region III will behave

almost classically as long as its total energy is larger than the local potential. As a quantum particle it will 'spend' complex time traversing those regions where the potential exceeds the energy (between the would-be turning points). During that 'time' it can be treated as a bounded particle in region II in an inverted potential $V(q) \to -V(q)$. After the barrier is left behind, the particle moves again almost classically in real time. The following boundary conditions have to be imposed:

$$\begin{aligned}
&\text{region I:} \quad q(0) = q_{\text{in}} \qquad q(t_1^\star) = q_1^\star \qquad (\Leftarrow p(t_1^\star) = 0) \\
&\text{region II:} \quad \tilde{q}(t_1^\star) = \tilde{q}_1^\star = q_1^\star \qquad \tilde{q}(t_2^\star) = \tilde{q}_2^\star = q_2^\star \\
&\text{region III:} \quad q(T) = q_{\text{fi}} \qquad q(t_2^\star) = q_2^\star \qquad (\Leftarrow p(t_2^\star) = 0)\,.
\end{aligned} \tag{13}$$

The tilde marks the classical forbidden configuration.

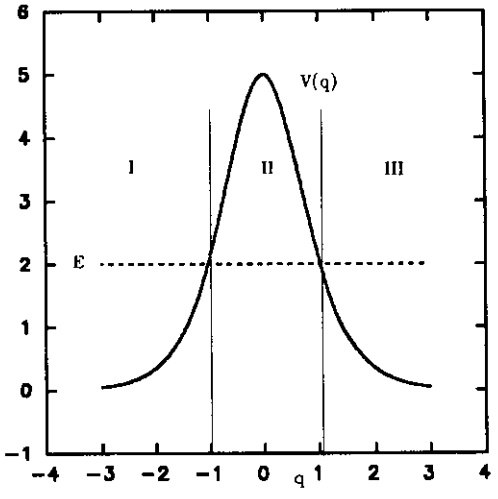

Figure 1. Potential barrier $V(q)$ together with energy E of a quantum particle.

What does it mean to minimize Γ when time is allowed to become complex? It is simply both real and imaginary parts of Γ which have to be minimized simultaneously, while the particle with energy E has to move in space from q_{in} to q_{fi}. The corresponding reduced action to be minimized is

$$\Gamma(q_{\text{fi}}, q_{\text{in}}; E) = ET + S(q_{\text{fi}}, q_{\text{in}}; T) \tag{14}$$

where T is the (supposed) time the particle needs to go from q_{in} to q_{fi}. Discretizing the time ($\Delta t_i = (t_{i+1} - t_i)$) and setting $q_i = q(t_i)$ we can write the action $S(q_{\text{fi}}, q_{\text{in}}; T)$ as

$$S = \sum_{i=1}^{N-1} \frac{1}{2} \frac{(q_{i+1} - q_i)^2}{\Delta t_i} - \sum_{i=1}^{N} V(q_i)\,\Delta t_i\,. \tag{15}$$

(We set $m = 1$ throughout the paper.) In order to allow the system to choose its time path freely we write each time interval explicitly as a complex number:

$$\Delta t_i = \delta_i + \mathrm{i}\Delta_i\,. \tag{16}$$

This results in the following decomposition:

$$\Gamma = \mathrm{Re}(\Gamma) + \mathrm{i}\,\mathrm{Im}(\Gamma) \equiv \Gamma_{\mathrm{R}} + \mathrm{i}\Gamma_{\mathrm{I}}$$
$$= \left(E \sum_{i=1}^{N-1} \delta_i + \frac{1}{2} \sum_{i=1}^{N-1} \frac{(q_{i+1} - q_i)^2}{\delta_i^2 + \Delta_i^2} \delta_i - \frac{1}{2} \sum_{i=1}^{N} V(q_i)(\delta_i + \delta_{i-1}) \right)$$
$$- \mathrm{i} \left(-E \sum_{i=1}^{N-1} \Delta_i + \frac{1}{2} \sum_{i=1}^{N-1} \frac{(q_{i+1} - q_i)^2}{\delta_i^2 + \Delta_i^2} \Delta_i + \frac{1}{2} \sum_{i=1}^{N} V(q_i)(\Delta_i + \Delta_{i-1}) \right). \tag{17}$$

Note that $q_1 = q_{\mathrm{in}}$, $q_N = q_{\mathrm{fi}}$, $\Delta_N = \delta_N = 0$.

Minimizing Γ_{R} and Γ_{I} simultaneously with respect to q_i, δ_i and Δ_i requires us to calculate the gradients

$$\frac{\partial \Gamma_{\mathrm{R,I}}}{\partial \delta_k} \qquad \frac{\partial \Gamma_{\mathrm{R,I}}}{\partial \Delta_k} \qquad \frac{\partial \Gamma_{\mathrm{R,I}}}{\partial q_k}$$

and to look for the minimum of the scalar quantity

$$F(q_k, \delta_k, \Delta_k) = \sum_k \left(\frac{\partial \Gamma_{\mathrm{R}}}{\partial \delta_k} \right)^2 + \sum_k \left(\frac{\partial \Gamma_{\mathrm{R}}}{\partial \Delta_k} \right)^2 + \sum_k \left(\frac{\partial \Gamma_{\mathrm{R}}}{\partial q_k} \right)^2 + \sum_k \left(\frac{\partial \Gamma_{\mathrm{I}}}{\partial \delta_k} \right)^2$$
$$+ \sum_k \left(\frac{\partial \Gamma_{\mathrm{I}}}{\partial \Delta_k} \right)^2 + \sum_k \left(\frac{\partial \Gamma_{\mathrm{I}}}{\partial q_k} \right)^2 \Longrightarrow \text{minimum}. \tag{18}$$

Our requirement that both the real and imaginary parts of the exponent Γ should be extremized separately, is due to the fact that the real part governs the phase of the amplitude (stemming from the Minkowskian domain of the path) and has to be stationary to avoid rapid cancellations, whereas the imaginary part determines directly the modulus (describing the Euclidean part of the path).

Now we turn to the problem of the calculation of the determinant. $\tilde{D}$ as given in (8) is obtained from the steepest descent paths and the corresponding Gaussian saddle-point integration over T. This suggests the inclusion of the determinant in the extremizing procedure with respect to (δ_i and Δ_i). Instead we adopt the approximation that the E-dependence of the determinant is specified through the complex time path found in (18). We write the first of the equations (8) in the form

$$\tilde{D} = \tilde{W} \left(-\frac{\partial^2 \Gamma}{\partial E^2} \right). \tag{19}$$

making use of the equality

$$\frac{\partial^2 S_{\mathrm{cl}}}{\partial T^2} = \left(-\frac{\partial^2 \Gamma}{\partial E^2} \right)^{-1}. \tag{20}$$

There is a method to compute $\tilde{W}$ to one-loop accuracy presented in [9] which is suitable in this case, too. Defining

$$\eta = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \tag{21}$$

we can rewrite $\tilde{W}$ as

$$\tilde{W} = \lim_{N\to\infty}\left(\frac{1}{2\pi i\hbar\epsilon}\right)^{N+1/2}\int \mathrm{d}^N\eta\,\exp(-\eta^T\sigma\eta)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2\pi i\hbar\epsilon}\,\frac{1}{(2i\hbar\epsilon)^N\det\sigma}\right)^{\frac{1}{2}} \tag{22}$$

where $\epsilon = T/(N+1)$. The matrix σ is the second-order derivative of the action taken at the classical path

$$\sigma_{ik} = \frac{1}{2}\frac{\partial^2 S}{\partial q_i \partial q_k}\bigg|_{q=q_{\mathrm{cl}}}. \tag{23}$$

For the case in hand we have to split the action along the path into Minkowskian and Euclidean parts with the correct boundary conditions. This results in the following form for σ:

$$\sigma = \frac{1}{2\epsilon\hbar i}\begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & & & & & & & \\
-1 & 2 & -1 & 0 & \cdots & & & & & & & \\
0 & -1 & 2 & -1 & 0 & \cdots & & & & & & \\
\ddots & \ddots & \ddots & \ddots & \cdots & & & & & & & \\
 & \cdots & 0 & -1 & 2 & -1 & 0 & \cdots & & & & \\
k^*: & & \cdots & 0 & -1 & 1+i & -i & 0 & \cdots & & & \\
 & & & \cdots & 0 & -i & 2i & -i & 0 & \cdots & & \\
 & & & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & & \\
 & & & & \cdots & 0 & -i & 2i & -i & 0 & \cdots & \\
k^{**}: & & & & & \cdots & 0 & -i & 1+i & -1 & 0 & \cdots \\
 & & & & & & \cdots & 0 & -1 & 2 & -1 & 0 \\
 & & & & & & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & & \cdots & 0 & -1 & 2 & -1 \\
 & & & & & & & & \cdots & 0 & -1 & 2
\end{pmatrix}$$

$$+\frac{i\epsilon}{2\hbar}\begin{pmatrix}
c_1 & 0 & \cdots & & & & & & & & \\
0 & c_2 & 0 & & \cdots & & & & & & \\
 & \cdots & \ddots & & \cdots & & & & & & \\
 & \cdots & 0 & \frac{1}{2}(1-i)c_{k^*} & & 0 & \cdots & & & & \\
 & & \cdots & 0 & & -i\,c_{k^*+1} & 0 & \cdots & & & \\
 & & & \cdots & & \ddots & \cdots & & & & \\
 & & & & & & \cdots & 0 & \frac{1}{2}(1-i)c_{k^{**}} & \cdots & \\
 & & & & & & & & \cdots & \ddots & \cdots \\
 & & & & & & & & \cdots & 0 & c_N
\end{pmatrix}. \tag{24}$$

c_k are defined by the relation

$$c_k = \frac{\partial^2 V}{\partial q^2}\bigg|_{q=q_{\mathrm{cl}}(t_k)}. \tag{25}$$

k^* and k^{**} label the boundaries between the Minkowski (I, III) and Euclidean (II) regions.

Now we introduce the notation p_N for the determinant,

$$p_N = (2i\hbar\epsilon)^N \det \sigma \,. \tag{26}$$

By expanding the minors of σ_j (the part of σ containing the first j columns and rows) one recognizes the following recursion formulae for $p_j = r_j + i\,t_j$.

Region I:

$$\begin{aligned} &r_{k+1} = (2 - \epsilon^2 c_{k+1}) r_k - r_{k-1} \qquad r_1 = 2 - \epsilon^2 c_1 \\ &r_0 = 1 \qquad t_{k+1} = (2 - \epsilon^2 c_{k+1}) t_k - t_{k-1} \qquad t_1 = t_0 = 0 \end{aligned} \tag{27}$$

Boundary I/II:

$$\begin{aligned} r_{k^*} &= (1 - \epsilon^2 c_{k^*}) r_{k^*-1} - r_{k^*-2} - (1 + \epsilon^2 c_{k^*}) t_{k^*-1} \\ t_{k^*} &= (1 - \epsilon^2 c_{k^*}) t_{k^*-1} - t_{k^*-2} + (1 + \epsilon^2 c_{k^*}) r_{k^*-1} \end{aligned} \tag{28}$$

Region II:

$$r_{k+1} = -(2 + \epsilon^2 c_{k+1})\, t_k + r_{k-1} \qquad t_{k+1} = (2 + \epsilon^2 c_{k+1})\, r_k + t_{k-1} \tag{29}$$

Boundary II/III:

$$\begin{aligned} r_{k^{**}} &= (1 - \epsilon^2 c_{k^{**}}) r_{k^{**}-1} + r_{k^{**}-2} - (1 + \epsilon^2 c_{k^{**}}) t_{k^{**}-1} \\ t_{k^{**}} &= (1 - \epsilon^2 c_{k^{**}}) t_{k^{**}-1} + t_{k^{**}-2} + (1 + \epsilon^2 c_{k^{**}}) r_{k^{**}-1} \end{aligned} \tag{30}$$

Region III:

$$r_{k+1} = (2 - \epsilon^2 c_{k+1}) r_k - r_{k-1} \qquad t_{k+1} = (2 - \epsilon^2 c_{k+1}) t_k - t_{k-1} \,. \tag{31}$$

These recursion relations are convenient to compute the one-loop determinant, especially in a numerical treatment where the limit $N \to \infty$ cannot be carried out practically. They give a complex result as it is expected in a tunnelling process.

The second factor in the approximation for $\tilde{D}$—$(\partial^2\Gamma/\partial E^2)$—can be computed in a straightforward manner once the E-dependence is obtained via (18).

3. An example

We illustrate our method for the Eckart potential

$$V(q) = \frac{\alpha}{\cosh^2(q)} \,. \tag{32}$$

The transmission coefficient of this potential is analytically known from a direct solution of the Schrödinger equation [10]. Miller [5] has discussed this problem within his complex path method up to the leading WKB approximation. Here we want to find the complex time path variationally. As a minimizing algorithm we use a conjugate-gradient method. Therefore, we need a starting configuration that

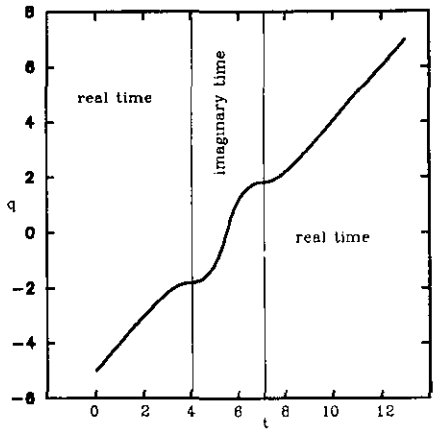

Figure 2. Space trajectory q for $E = 0.5$. Time t is partly real, partly imaginary (in the same units).

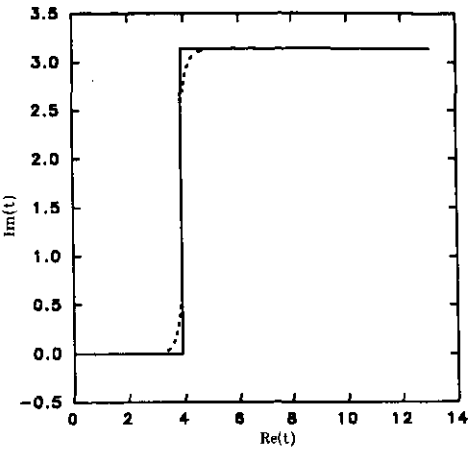

Figure 3. Initial (broken curve) and final (full curve) time path in the complex plane, corresponding to $E = 0.5$ (figure 2).

should not be too far away from the real extremum. As input we use the separate solutions of the classical equations of motion in the three regions (in real or imaginary time, respectively) glued together at the boundaries in accordance with the boundary conditions (13). These solutions have been constructed with a Runge–Kutta method with a 'time' step size of $h = 0.02$. But additionally we smeared out the edges of the complex time path. For the following four example runs we took the parameters:

set 1: $\alpha = 5$ $E = 4.5$ $q_{\text{in}} = -5$ $q_{\text{fi}} = 7$ $N_{\text{osc}} = 0$ (33)

set 2: $\alpha = 5$ $E = 0.5$ $q_{\text{in}} = -5$ $q_{\text{fi}} = 7$ $N_{\text{osc}} = 0$ (34)

set 3: $\alpha = 5$ $E = 0.1$ $q_{\text{in}} = -5$ $q_{\text{fi}} = 7$ $N_{\text{osc}} = 0$ (35)

set 4: $\alpha = 5$ $E = 4.5$ $q_{\text{in}} = -5$ $q_{\text{fi}} = 7$ $N_{\text{osc}} = 1$ (36)

where N_{osc} denotes the number of oscillations between the turning points in region II. In figure 2 the space trajectory q is shown versus time for set 2. The regions of real and imaginary time increments, respectively, are indicated. No visible difference could be recognized between the trajectories before and after conjugate gradient minimization, i.e. the 'classical' solution is very near to the final configuration. Perhaps also interesting are the initial and final complex time paths, given in figure 3. One can see that the system prefers a sharp edge from the real to the purely imaginary part of the path and vice versa. It should be noticed that the total imaginary time interval amounts to π, which corresponds precisely to the imaginary spacing of branching points in (12). Figure 4 shows a space trajectory versus time for the set 4 (energy $E = 4.5$) which oscillates back and forth before leaving the barrier as a right mover. The energy had been chosen such that the velocity v was exactly three times as large as in set 2. Therefore, due to the additional oscillation below the barrier, the imaginary time step is again π and the time path looks practically as in figure 3. Quantitatively we have to report the following reduced action values and variation functionals before

and after conjugate gradient minimization, together with the fluctuation determinants for the respective energies:

Set 1
$$\Gamma_I^{inital} = 0.5132 \qquad \Gamma_I^{final} = 0.5097$$
$$\Gamma_R^{inital} = 771.8 \qquad \Gamma_R^{final} = 769.3$$
$$F^{inital} = 0.765 \qquad F^{final} = 0.784 \times 10^{-5}$$
$$p_N = 1.01 + i\,0.044 = 1.01\,e^{i\,0.04}$$

Set 2
$$\Gamma_I^{inital} = 6.797 \qquad \Gamma_I^{final} = 6.793$$
$$\Gamma_R^{inital} = 359.4 \qquad \Gamma_R^{final} = 354.9$$
$$F^{inital} = 0.949 \qquad F^{final} = 0.794 \times 10^{-5}$$
$$p_N = 48.01 + i\,0.145 = 48.03\,e^{i\,0.003}$$

Set 3
$$\Gamma_I^{inital} = 8.529 \qquad \Gamma_I^{final} = 8.529$$
$$\Gamma_R^{inital} = 277.4 \qquad \Gamma_R^{final} = 277.4$$
$$F^{inital} = 0.1 \times 10^{-2} \qquad F^{final} = 0.654 \times 10^{-5}$$
$$p_N = 283.4 + i\,4.137 = 300.5\,e^{i\,0.015}$$

Set 4
$$\Gamma_I^{inital} = 1.562 \qquad \Gamma_I^{final} = 1.531$$
$$\Gamma_R^{inital} = 811.7 \qquad \Gamma_R^{final} = 784.7$$
$$F^{inital} = 1.91 \times 10^{1} \qquad F^{final} = 0.717 \times 10^{-5}$$
$$p_N = 0.0337 + i\,0.943 = 0.9449\,e^{i\,1.537}.$$

(37)

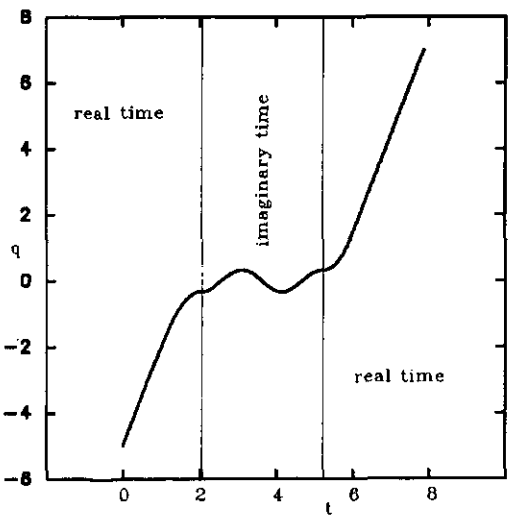

Figure 4. Space trajectory q for $E = 4.5$, performing one oscillation under the barrier. Time t is partly real, partly imaginary (in the same units).

4. Discussion

The quantum mechanical toy model investigated shows a behaviour with surprisingly clean separation between the three regions, with a sharp-edged complex time path. This result was not obvious from the beginning, at least to us. To our knowledge a variational method to obtain explicitly space trajectory and time path has not appeared before in the literature. Moreover, we applied the well known method for calculating the determinant of fluctuations on top of the leading trajectory (in each topological sector) to the case in hand, where it handles each trajectory uniformly over its real and imaginary time sections. The justification to use the recursion relations in their form (27)–(31) is provided by the 'rectangular' transition from the Minkowski to the Euclidean time regime.

Our choice of for the potential $V(q)$ was determined by the possibility to compare our result with the exact solution known for a long time [10]. The transmission coefficient is given by

$$D(E)_{\text{exact}} = \frac{\sinh^2 \pi k}{\sinh^2 \pi k + \cosh^2\left(\frac{1}{2}\pi\sqrt{8\alpha - 1}\right)} \tag{38}$$

with k defined by $k = \sqrt{2E}$. In our approach the transmission coefficient $D(E)$ has the form

$$D_{N_{\text{osc}}} = \tilde{D}_{N_{\text{osc}}} \tilde{D}^{*}_{N_{\text{osc}}} \tag{39}$$

with $\tilde{D}_{N_{\text{osc}}}$ defined by

$$\tilde{D}_{N_{\text{osc}}}(E) = \frac{1}{\sqrt{2\pi}} e^{i\Gamma_R(E)} \sum_{i=0}^{N_{\text{osc}}} \frac{1}{\sqrt{p_N^{(i)}(E)\partial^2\Gamma^{(i)}/\partial E^2}} e^{-\Gamma_I^{(i)}(E)}. \tag{40}$$

The example runs for the sets 1, 2 and 3 compared to the exact result as shown in table 1.

Table 1.

E	$D(E)_{\text{exact}}$	$D_0(E)$
0.1	4.4×10^{-8}	3.5×10^{-9}
0.5	1.6×10^{-6}	1.9×10^{-6}
4.5	0.316	0.316

We note that we have done the path integral and the integration over T within the Gaussian saddle-point approximation. A non-perturbative path integration, which is possible in the Euclidean region, could improve our result. So far, the numerical value for $E = 4.5$ has been fitted to the exact result in order to compensate for these approximations. Then our results show a deviation from $D(E)_{\text{exact}}$ at smaller energy E. Nevertheless, the agreement is not bad. For set 4 we have the following formula for the transmission coefficient:

$$\tilde{D}_1(E) = \frac{1}{\sqrt{2\pi\partial^2\Gamma/\partial E^2}} e^{i\Gamma_R(E)} e^{-\frac{1}{2}i\phi_N^{(0)}} \times \left(\frac{1}{\sqrt{u_N^{(0)}}} e^{-\Gamma_I^{(0)}(E)} + \frac{1}{\sqrt{u_N^{(1)}}} e^{-\frac{1}{2}i\left(\phi_N^{(1)} - \phi_N^{(0)}\right)} e^{-\Gamma_I^{(1)}(E)} \right) \tag{41}$$

where we have set $p_N(E) = u_N e^{i\phi_N}$ and used the (numerically found) approximative equality

$$\frac{\partial^2\Gamma^{(0)}}{\partial E^2} \approx \frac{\partial^2\Gamma^{(1)}}{\partial E^2} = \frac{\partial^2\Gamma}{\partial E^2}.$$

From the numbers in (28) we find for phase difference in (29)

$$\Delta\phi = \phi_N^{(1)} - \phi_N^{(0)} = 1.497$$

which is rather close to the expected value $\Delta\phi = \pi/2$ [9]. This demonstrates that our method for calculation of the one-loop determinant also reproduces the relative phases.

McLaughlin [4] has also used complex time variables and got similar results especially for the resulting reduced action. Carlitz and Nicole [6] calculated the transition probability for various quantum mechanical examples including parabolic potential barriers. They considered all possible saddle points and the corresponding steepest descent paths in the complex time plane. But explicit trajectories were not available within their approach.

It was our intention to demonstrate the complex time path as it results from the variational principle (18). The moral fits perfectly into the discussion concerning the periodic instanton approach [3] . There, the detailed form of the complex time path is crucial for the method. However, it is known that results of quantum mechanics cannot be straightforwardly taken over into quantum field theory. The main obstacle lies in the fact that a classical turning point can hardly be defined for a system with many degrees of freedom. It remains to be seen how the variational principle (18) copes with a problem with a few degrees of freedom. It is merely the use of solutions of the equations of motion (in Minkowski or Euclidean time), which needs the existence of the turning points as prerequisite.

A possible way out in order to keep this heuristic tool could be the truncation of the many degrees of freedom to effectively a single one (reaction coordinate) and a description of the system in terms of (quantum) effective mass and potential. Taken as the final answer, this would possibly imply an uncontrollable loss of physical reality. A first step in this direction was taken by Bitar and Chang [11] more than a decade ago. They were the first to address vacuum tunnelling in a quantum field theory in Minkowski space. In order to do this they interpreted the Euclidean time of the instanton as a function $\lambda(t)$ of a typically real, occasionally imaginary time parameter. Within the standard ansatz for the pseudoparticle solution and the integration over 3-space they obtained an effective theory with one degree of freedom (with completely classical mass and potential at this level). Although their ansatz is not applicable for the combined Higgs–Yang–Mills system the general idea could be of interest in combination with our toy model discussed above. A partly real, partly imaginary time trajectory obtained through this truncation could, at least, serve as input for the numerical variational problem. It should be noted, however, that it is difficult to extend the class of ansätze taken as input. One should at least go beyond the O(4)-symmetry in order to be much more realistic than using other approaches.

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